



Activity



- Fibonacci Spiral -

Step 1: Introduction

Look closely at the following image:



It is a pine cone. We notice that the scales of the pine cone form a spiral shape. If we count the number of spirals (as done on the image), we find that there are 8 clockwise spirals and 13 counterclockwise spirals.

Try to find a pine cone or a pineapple. With a marking pen, draw the spirals created by the scales (as done on the drawing). Count how many spirals there are.

Step 2: The Fibonacci sequence

Here is the beginning of the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21,...

Do you notice something?

Each Fibonacci number is the result of the addition of the two previous numbers.

Why is it an interesting sequence?

The Fibonacci numbers can be found, among others, in nature. Look carefully at the pine cone in step 1. What do you notice? That's right! The number of spirals, in either directions, is part of the Fibonacci sequence! If you used your own pine cone or a pineapple to count the spirals, you should also have found Fibonacci numbers.

There are other reasons why this sequence fascinates mathematicians and the public. It has several interesting characteristics. Here are two of them:

A- *The squares' sum of the n first terms is equal to the product of the terms in position n and $n+1$.*

Examples :

$$1^2+1^2+2^2+3^2=1+1+4+9 = 15 = 3*5$$

$$1^2+1^2+2^2+3^2+5^2=1+1+4+9+25 = 40 = 5*8$$

Why?

It is in fact two different expressions for the area of a same rectangle. The big side of this rectangle is the $(n+1)^{\text{th}}$ term of the sequence.

To better visualize this, use the Geogebra application on the Fibonacci sequence by following this internet link: <https://www.geogebra.org/m/XnUkTse6>. Move the cursor¹ up to the value 5. We notice that the drawn squares all have numbers of the Fibonacci sequence as their side measurements.

¹ The cursor is at the bottom of the Geogebra page. It is shaped like a dot that we can move along a line. By moving that dot, we change the value of "a" and that progressively forms the drawing.

Plus, the biggest square has a side of 5 units. In other words, the sequence stops at the term 5. The “squares’ sum of the n first terms” is, in this case, $1^2+1^2+2^2+3^2+5^2$, which equals $1+1+4+9+25$, so 40. This sum is also the area’s sum of all the squares together.

Together, all the squares that we see on the screen form a rectangle that has a side of 5 units and a side of 8 units (8 is precisely the term that follows 5 in the Fibonacci sequence). If we wanted to calculate the area of this rectangle, we could add the area of the squares that form it (we would find 40, because it is precisely the addition done above). We could also directly calculate the area by doing $8*5 = 40$.

So, we just “geometrically” proved that no matter where we are in the Fibonacci sequence, there is an equality between, on one hand, the sum of the n first terms of the sequence, squared, and, on the other hand, the product of n^{th} and the $(n+1)^{\text{th}}$ terms.

B- If we divide two consecutive terms: the farther we are in the sequence, the closer the result gets to the golden ratio (φ).

Do the following divisions and note the results: $2/1$, $3/2$, $5/3$, $8/5$, $13/8$, $21/13$, $34/21$.

What do you notice?

The results are closer and closer to 1,6. In fact, the results are getting closer to the golden ratio. The golden ratio is represented by φ and its value is approximately 1,618. The golden ratio would be the perfect ratio. We see it in many artworks.

An example of the presence of the golden ratio in geometry and in drawings is the star in a regular pentagon. Indeed, we observe that the ratio of the star’s side on the pentagon’s side corresponds the the golden ratio. To see it, use the Geogebra document “The pentagon, its star and the golden ratio” (you can access it by following this internet link:). You can move the P dot as much as you want, which will change the pentagon’s dimensions, but not the ratio we are interested in, as you will notice.

Step 3: The Fibonacci spiral

From the Fibonacci sequence, we can draw the Fibonacci spiral. You can visualize it on the “Fibonacci Spiral” Geogebra document (<https://www.geogebra.org/m/XnUkTse6>) by moving the cursor up to its maximum value. Between the values 6 and 13 of the cursor, we see the Fibonacci spiral being drawn. This spiral is said to be “perfect”, particularly because of its link to the golden ratio (φ). In nature, the petals or the leaves often position themselves in a way to optimize the space, and one of the layouts involves an angle of $[360^\circ - (360/\varphi)^\circ]$ between each element. It is this layout that creates the spiral design that we saw on the pine cone in step 1 and that reminds us of the Fibonacci spiral.