PUZZLING CARTOON

-STAINED GLASS-

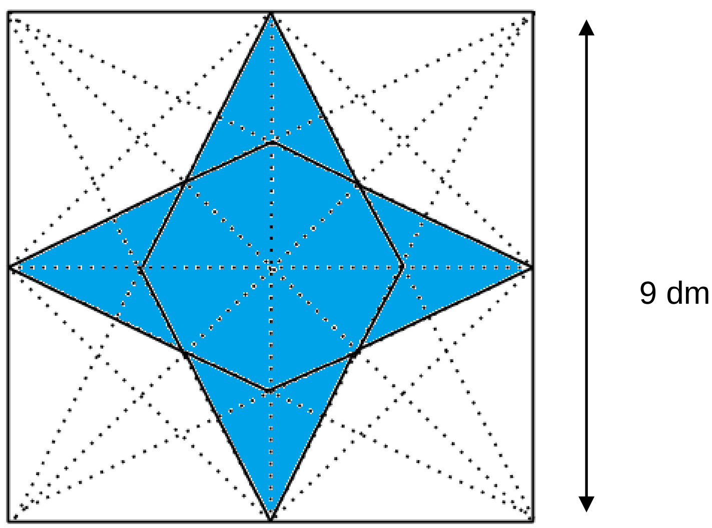
AMAZINGMATHS

Materials:

- Video of the puzzle
- Sheets of paper
- Pencils

The puzzle

To underline the orientation service of the school, the principal orders a stained-glass window representing a wind rose. In this stained glass, the dotted lines link up either the middle points of the sides or the corners of the square. The sides of the stained glass measure 9 dm.



To help the principal with the measurements, can you determine the area of the blue region?

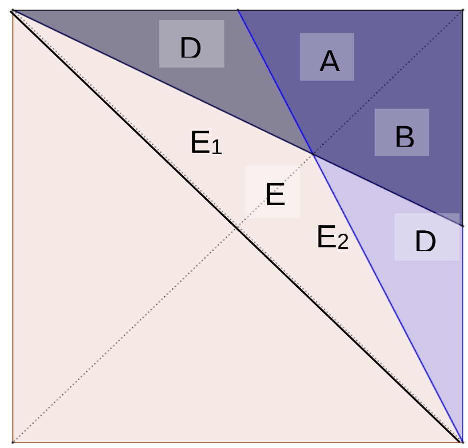
PUZZLE SOLUTION

The answer:

The area of the blue region is 27 dm^2. There are several ways to solve this puzzle. We explain 4 in the following pages, but it is absolutely possible for your students or yourself to find the right answer using another method!

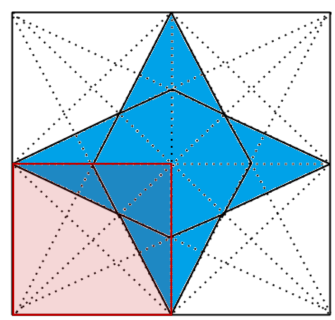
First solution

Since the shape is fairly complicated, let's begin by observing a part of it. Let's look only at the square below, which represents the fourth of the figure.



The grey triangle's area corresponds to a fourth of this square, because the triangle's hypotenuse links the corner to the middle of the right side. For the same reason, the blue triangle's area is also equivalent to a fourth of the square. If the two triangles were not superposed onto the regions called A and B, then they would cover half of this square.
The difference of area between the square's half and the covered region is then given by the areas of the triangles A and B.
Plus, it is also given by the area of the big triangle E. Triangles A, B, E_1 and E_2 therefore have equivalent areas.
Triangles D_1 and D_2 also have the same area since they have the same base and the same height as A and B.

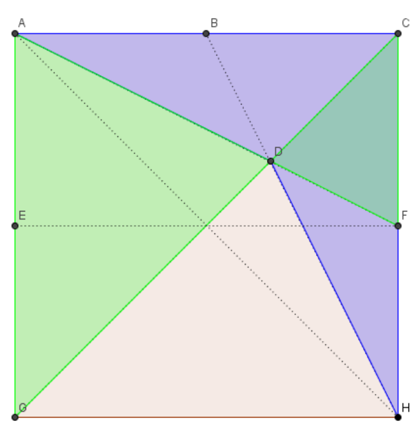
All the triangles of the square's upper half then have the same area. Since 4 of the six triangles are part of the stained glass, that is 2/3 of the area of the square's half, we find that the total area of the blue region on the stained glass is:



$$A_{blue} = \frac{4}{6} * \frac{1}{2} * A_{square} = \frac{2}{3} * \frac{1}{2} * c^2 = \frac{1}{3} * c^2$$

We can apply the result to the whole stained glass, because all the quarters are identical.
For the big square, *c* measures 9 dm and the blue region's area is **27 dm^2.**

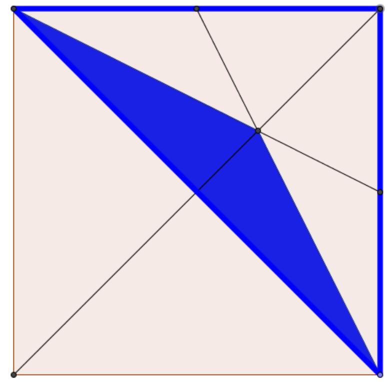
Second solution



The two green triangles have a ratio of similitude of ½. Indeed, they are similar by AA (opposite angles and internal and external angles) and their equal sides supported by the triangle have a ratio of 1:2. Their areas then have a ratio of 1:4. The small blue triangle, DFH, has the same area as DCF because they have the same height and congruent bases. Therefore, together, triangles DCF and DFH have an area equal to the half of ADG's area. By symmetry, we can say that ACD has an area equal to the half of ADG. Thus, triangle ACD occupies a third of triangle ACG and triangle DCH occupies a third of triangle GCH. The part in blue represents a third of the square and, if we consider the whole stained glass, the blue region occupies a third of the total area of the big square. We then obtain $A_{blue} = A_{square} * \frac{1}{3} = 9 * 9 * \frac{1}{3} = \frac{81}{3} =$ **27 dm²**.

Third solution

Theorem: an isosceles triangle's medians cross at a third of its height.

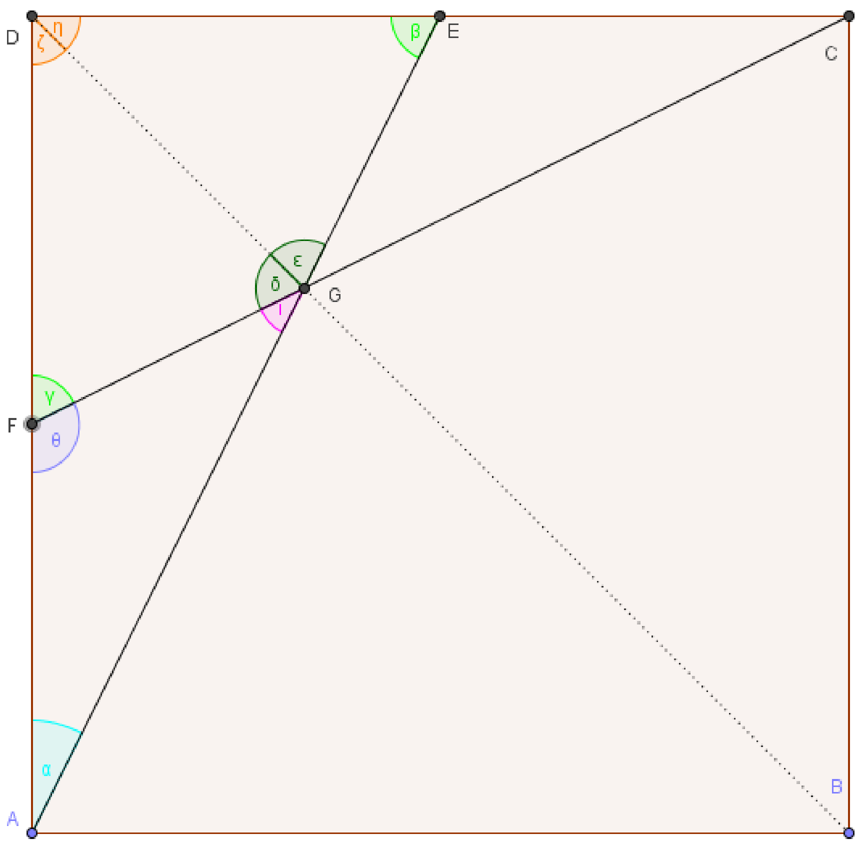


Here, the dark blue triangle has the same base as the blue edge triangle. The height of the smallest triangle is a third of the height of the big triangle according to the theorem above.
The difference between the blue edge triangle and the dark blue triangle corresponds to the light blue region on the part of the initial stained glass that we consider. So, this region occupies 2/3 of the square's half. We can generalize to the whole stained glass and state that the stained glass' area we are searching for is

$$A_{blue} = \frac{2}{3} * \frac{1}{2} * A_{square} = \frac{2}{3} * \frac{1}{2} * 9 * 9 = \mathbf{27\ dm^2}.$$

Fourth solution

<u>Lower right quarter of the stained glass.</u>



On the figure above, all the angles of the same colour have the same measurement. We get to that result by symmetry.
We are considering the right triangle ADE. Since the square of the figure corresponds to a fourth of the stained glass, the triangle has a first leg of 2,25 dm and the other one measures 4,5 dm. The hypotenuse is then $\sqrt{(2{,}25)^2 + (4{,}5)^2} = \sqrt{25{,}3125} \approx 5{,}03114295$.
We can find the value of angle α with $\arcsin(\frac{2{,}25}{\sqrt{25{,}3125}})$ = 26,565º.

We can find the value of angle β with $\arcsin(\frac{4{,}5}{\sqrt{25{,}3125}})$=63,43495º or with 90º – α.

On the figure, we see that:
$\beta + \varepsilon + \eta = 180^0$ where η=45º
We then find that $\varepsilon = 180 - 45 - 63{,}43495 = 71{,}56505^0$.

We find the measurements of triangle DEG with the Sine law:

$$\frac{a}{\sin A} = \frac{b}{sin\ B} = \frac{c}{\sin C}$$

We know that DE measures 2,25 dm, so:

$$\frac{2{,}25}{\sin 71{,}56505^0} = \frac{DG}{sin\ 63{,}43495^0} = \frac{EG}{\sin 45^0}$$

Which gives DG=2,12132 dm and EG=1,67705 dm.

In a homologous way, we can find the length of triangle AFG's sides.
EG=FG = 1,67705 dm
AF=2.25 dm

$$\frac{1{,}67705}{\sin 26{,}565^0} = \frac{AG}{sin\ (180^0 - 63{,}43495^0)}$$

AG=3,35411 dm

We can then find the area of triangle ADG with Heron's formula (or with another method).
Heron's formula: $A = \sqrt{p(p-a)(p-b)(p-c)}$
Where *p* is the semiperimeter $\frac{a+b+c}{2}$
Here *p* is worth: $p = \frac{4{,}5+2{,}12132+3{,}35411}{2}$=4,987715.

$$A = \sqrt{4{,}987715(4{,}987715 - 4{,}5)(4{,}987715 - 2{,}12132)(4{,}987715 - 3{,}35411)} = 3.375012744$$

Since triangle ADG corresponds to the eighth of the area of the stained glass' blue region, we multiply the obtained result by 8.

$$A = 8 * 3.375012744 = 27.00010195$$

Since we rounded off, we obtain a solution that is almost the right one: **27 dm^2**.